RAMAKRISHNA MISSION VIDYAMANDIRA

(A Residential Autonomous College under University of Calcutta)

First Year

First-Semester Examination, December 2010

Date : 23-12-2010	**MATHEMATICS (General)**	Full Marks : 75
Time : 11am – 2pm	**Paper - I**	

(Use separate answer script for each group)

Group – A

Answer Question No. 1 and any two from the rest.

1. a) Answer any one question : [2]

   i) Find the values of $(-i)^{3/5}$.

   ii) Find the quotient polynomial and remainder when $x^4+5x^3+4x^2+8x-2$ is divided by $(x+2)$.

   b) Answer any one question : [3]

   i) Apply Descartes' Rule of signs to find the nature of the roots of the equation $x^4+2x^2-7x-5=0$

   ii) Without expanding the determinant prove that $\begin{vmatrix} 0 & (x-y)^3 & (x-z)^3 \\ (y-x)^3 & 0 & (y-z)^3 \\ (z-x)^3 & (z-y)^3 & 0 \end{vmatrix}=0$

2. a) Show that $\sin\left(i\log\dfrac{a-ib}{a+ib}\right)=\dfrac{2ab}{a^2+b^2}$, where a and b are real. [4]

   b) Prove that $i\log\dfrac{x-i}{x+i}=\pi-2\tan^{-1}x$, x is real. [3]

   c) Prove that $\sin(\log i^i)=-1$. [3]

3. a) Solve the equation $x^3-12x+65=0$ by cardan's method. [4]

   b) If α,β,γ be the roots of $x^3+px^2+qx+r=0$, form the equation whose roots are $\beta+\gamma-2\alpha$, $\gamma+\alpha-2\beta$, $\alpha+\beta-2\gamma$ and hence find the value of $(\beta+\gamma-2\alpha)(\gamma+\alpha-2\beta)(\alpha+\beta-2\gamma)$. [4+2]

4. a) Show, without expanding, that $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ b+c & c+a & a+b \end{vmatrix}=(b-c)(c-a)(a-b)(a+b+c)$ [5]

   b) Solve by Cramer's rule, [5]
   $$\begin{aligned} x-y+2z&=1\\ x+y+z&=2\\ 2x-y+z&=5 \end{aligned}$$

5. a) Find the adjoint and inverse of the matrix $\begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & 0 \\ 2 & 1 & 1 \end{pmatrix}$. [2+2]

b) Find the rank of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$. [3]

c) Show that the matrix $A = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ -2 & -2 & -1 \end{pmatrix}$ is orthogonal. [3]

<u>Group – B</u>

Answer Question No. 6 and <u>any two</u> from the rest.

6. a) Answer <u>any one</u> question : [2]

   i) If A = {1,2,3}, B = {3,4,5,6}, C = {1,2,5,7}; find $A \cup (B - C)$.

   ii) If $f : R \to R$ and $g : R \to R$ be two mappings given by $f(x) = x^2 + 3x + 1$ and $g(x) = 2x - 3$ find (fog)(x)

   b) Answer <u>any one</u> question : [3]

   i) In a group $(G, *)$ show that $(a * b)^{-1} = b^{-1} * a^{-1} \ \forall \ a, b \in G$.

   ii) A mapping $f : R \to R$ is defined by

   $f(x) = \begin{cases} x^2 - 1, & \text{if } x \geq 0 \\ -x^2 - 1, & \text{if } x \leq 0 \end{cases}$. Determine whether the mapping is bijective.

7. a) Prove that if $f : A \to B$, $g : B \to C$ are two bijective maps then $gof : A \to C$ is also bijective. [4]

   b) Show that the set of all non-zero real numbers forms a group with respect to multiplication. [4]

   c) If in a group $(G, *)$, $(a * b)^{-1} = a^{-1} * b^{-1} \ \forall \ a, b \in G$; show that G is abelian. [2]

8. a) Let $(G, *)$ be a group and a, b $\in G$. Prove that the equation $a * x = b$ has a unique solution in G. [4]

   b) Prove that the set $T = \left\{ \frac{p}{q} : p, q \text{ are odd integers} \right\}$ is a group with respect to multiplication. [4]

   c) In a ring (R, +, ·), prove that a. (–b) = –(a.b) $\forall \ a, b \in R$ [2]

9. a) Define a subspace of a vector space. Prove that the subset S of R^3 defined by

   $S = \left\{ (x, y, z) \in R^3 : y = z = 0 \right\}$ is a subspace of the vector space R^3 over R. [2+3]

   b) Prove that the set of vectors {(1, 2, 2), (2, 1, 2), (2, 2, 1)} forms a basis of the vector space R^3 over R. [5]

10. a) Find the eigen values of the matrix $\begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{pmatrix}$. Hence find the eigen vector corresponding to the negative eigen value. [3+2=5]

   b) State Cayley Hamilton theorem. Use it to find A^{-1} of the matrix $A = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{pmatrix}$. [1+4 = 5]

Group – C

Answer Question No. 11 and any two from the rest. [25]

11. a) Answer any one question : [2]

i) Find the domain of definition of the function : $f(x) = \dfrac{1}{\sqrt{|x| - x}}$.

ii) If y = sin2x, find $(y_5)_0$.

b) Answer any one question : [3]

i) Show that for the curve $r = ae^{\theta \cot \alpha}$ (α-constant), the angle between the tangent at any point P on the curve and the radius vector OP is always constant, O being the pole.

ii) Prove or disprove : If f be continuous at a point c of its domain, then f' exists at that point.

12. a) If $f(x) = 2|x| + |x - 2|$, find $f'(1)$. [5]

b) Show that the pedal equation of $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ w.r.t. origin is $\dfrac{a^2b^2}{p^2} = a^2 + b^2 - r^2$. [5]

13. a) If $y = \sin(m \sin^{-1}x)$, show that $(1 - x^2)y_2 - xy_1 + m^2y = 0$ and hence prove that $(1 - x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 - m^2)y_n = 0$ [5]

b) Given, $f(x, y) = xy.\dfrac{x^2 - y^2}{x^2 + y^2}$, if $x^2 + y^2 \neq 0$

$= 0$, if $x^2 + y^2 = 0$

show that $f_{xy}(0,0) \neq f_{yx}(0,0)$. [5]

14. a) Find $\dfrac{dy}{dx}$ where $(\cos x)^y = (\sin y)^x$. [2]

b) If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ show that, $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = \dfrac{3}{x + y + z}$ [4]

c) If ρ, ρ' be the radii of curvature at the ends of two conjugate diameters of an ellipse, prove that—

$\left(\rho^{2/3} + \rho'^{2/3}\right)(ab)^{2/3} = a^2 + b^2$. [4]

15. a) If $p = x\cos\alpha + y\sin\alpha$ touches the curve $\left(\dfrac{x}{a}\right)^{\frac{n}{n-1}} + \left(\dfrac{y}{b}\right)^{\frac{n}{n-1}} = 1$, then prove that—

$p^n = (a\cos\alpha)^n + (b\sin\alpha)^n$. [5]

b) If $u = \tan^{-1}\dfrac{x^3 + y^3}{x - y}$ then prove that

i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$.

ii) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = (1 - 4\sin^2 u)\sin 2u$. [2+3]